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# UNBENDING SHAPES OF THIN-WALLED FLAT TRANSLATIONAL SHELLS OF VARIABLE THICKNESS

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*Consideration is given to the problem on selection of the thickness of a flat translational shell in which the prescribed external load and temperature field lead only to a zero-moment stressed-strained state (i.e., generate only membrane forces and do not change the curvature of the median surface). Within the framework of the Kirchhoff–Love theory, this problem is reduced to solution of a nonlinear differential equation.*

Shell structures (shells) are widely used as domes, ceilings, etc.; therefore, their analysis for strength represents a topical problem in the modern mechanics of a deformed body. In view of the mathematical complexity of this problem, one often simplifies it by making a number of assumptions (flatness of the shell, calculation and designing of the shell for a prescribed load according to the zero-moment theory with the edge effect imposed, and others).

We use the assumptions that the median surface is described by the following equations [1–4]:

$$z = f(x) + g(y) , \quad x \in [0; a] , \quad y \in [0; b] ;$$
$$|z'_x| << 1 , \quad |z'_y| << 1 , \quad A = B = 1 , \quad \frac{1}{R_1} \approx - f''(x) , \quad \frac{1}{R_2} \approx - g''(y) , \quad \frac{1}{R_{12}} = 0 . \tag{1}$$

The problem in question is in selecting the thickness of the shell $h(x, y)$ such that the prescribed external load $q_1$, $q_2$, and $q_n$ and temperature field $\theta$ in it produce no change in its curvature and no torsion, i.e.,

$$\chi_1 = \chi_2 = \chi_{12} = 0 . \tag{2}$$

Within the framework of the Kirchhoff–Love theory, the resolving equations of this problem take the following form [1]:

the equilibrium equations appear as

$$\frac{\partial T_1}{\partial x} + \frac{\partial S}{\partial y} + q_1 = 0 , \quad \frac{\partial T_2}{\partial y} + \frac{\partial S}{\partial x} + q_2 = 0 , \quad \frac{T_1}{R_1} + \frac{T_2}{R_2} = q_n ; \tag{3}$$

Hooke's law with allowance for the temperature strain is

$$\varepsilon_1 = \frac{1}{Eh}(T_1 - \mu T_2) + \alpha\theta , \quad \varepsilon_2 = \frac{1}{Eh}(T_2 - \mu T_1) + \alpha\theta , \quad \gamma_{12} = \frac{2(1+\mu)}{Eh} S ; \tag{4}$$

the equations of consistency of strains appear as

$$\frac{\partial \varepsilon_2}{\partial x} = \frac{\partial \gamma_{12}}{\partial y} , \quad \frac{\partial \varepsilon_1}{\partial y} = \frac{\partial \gamma_{12}}{\partial x} , \quad \frac{\partial^2 \gamma_{12}}{\partial x \partial y} = 0 . \tag{5}$$

From Eq. (5) we have

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$$\gamma_{12} = u(x) + v(y) . \tag{6}$$

Then we obtain

$$u(x) + v(y) = \frac{2(1+\mu)}{Eh(x, y)} S(x, y) \tag{7}$$

or

$$S(x, y) = \frac{Eh(x, y)}{2(1+\mu)} (u(x) + v(y)) . \tag{8}$$

The functions $u(x)$ and $v(y)$ involved in (7) and (8) are determined from the following conditions:

$$\begin{gathered} u(x) + v(0) = \frac{2(1+\mu)}{Eh(x, 0)} S(x, 0) , \quad u(0) + v(y) = \frac{2(1+\mu)}{Eh(y, 0)} S(0, y) , \\ u(0) + v(0) = \frac{2(1+\mu)}{Eh(0, 0)} S(0, 0) . \end{gathered} \tag{9}$$

Adding these equalities termwise, we obtain

$$\frac{S(x, y)}{h(x, y)} = \frac{S(x, 0)}{h(x, 0)} + \frac{S(0, y)}{h(0, y)} - \frac{S(0, 0)}{h(0, 0)} . \tag{10}$$

The boundary values of $S(x, y)$ and $h(x, y)$ are involved in the right-hand side of (10); therefore, $\frac{S(x, y)}{h(x, y)}$ may be considered to be known. Then (1) yields

$$\frac{\partial T_1}{\partial x} = -\left(\frac{\partial S}{\partial y} + q_1\right), \quad \frac{\partial T_2}{\partial y} = -\left(\frac{\partial S}{\partial x} + q_2\right). \tag{11}$$

From Eqs. (4) and (5) we obtain

$$\frac{\partial}{\partial y}\left(\frac{T_1 - \mu T_2}{h(x, y)} + E\alpha\theta\right) = \frac{\partial}{\partial x}\left(\frac{2(1+\mu)}{h(x, y)} S(x, y)\right), \quad \frac{\partial}{\partial x}\left(\frac{T_2 - \mu T_1}{h(x, y)} + E\alpha\theta\right) = \frac{\partial}{\partial y}\left(\frac{2(1+\mu)}{h(x, y)} S(x, y)\right),$$

whence we have

$$\begin{aligned} \frac{\partial T_1}{\partial y} &= -\mu\left(\frac{\partial S}{\partial x} + q_2\right) - \frac{\partial}{\partial y} \ln h\,(T_1 - \mu T_2) - h\frac{\partial}{\partial y}(E\alpha\theta) + h\frac{\partial}{\partial x}\left(\frac{2(1+\mu)}{h} S(x, y)\right), \\ \frac{\partial T_2}{\partial x} &= -\mu\left(\frac{\partial S}{\partial y} + q_1\right) - \frac{\partial}{\partial x} \ln h\,(T_2 - \mu T_1) - h\frac{\partial}{\partial x}(E\alpha\theta) + h\frac{\partial}{\partial y}\left(\frac{2(1+\mu)}{h} S(x, y)\right). \end{aligned} \tag{12}$$

Formulas (11)–(12) represent a system of linear differential equations with partial derivatives of first order for $\frac{\partial T_1}{\partial x}$, $\frac{\partial T_1}{\partial y}$, $\frac{\partial T_2}{\partial x}$, and $\frac{\partial T_2}{\partial y}$ whose solvability conditions have the form [5–7]

$$\frac{\partial}{\partial y}\left(\frac{\partial S}{\partial y}+q_1\right)=\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial S}{\partial x}+q_2\right)+\left(\frac{\partial}{\partial y}\ln h\right)(T_1-\mu T_2)+h\frac{\partial}{\partial y}(E\alpha\theta)-h\frac{\partial}{\partial x}\left(\frac{2\,(1+\mu)}{h}S\,(x,y)\right)\right),$$
$$\frac{\partial}{\partial x}\left(\frac{\partial S}{\partial x}+q_2\right)=\frac{\partial}{\partial y}\left(\mu\left(\frac{\partial S}{\partial y}+q_1\right)+\left(\frac{\partial}{\partial x}\ln h\right)(T_2-\mu T_1)+h\frac{\partial}{\partial x}(E\alpha\theta)-h\frac{\partial}{\partial y}\left(\frac{2\,(1+\mu)}{h}S\,(x,y)\right)\right). \tag{13}$$

Hence we find

$$\left(\frac{\partial^2}{\partial y\partial x}\ln h\right)(T_1-\mu T_2)+\frac{\partial}{\partial y}\ln h\left(\frac{\partial T_1}{\partial x}-\mu\frac{\partial T_2}{\partial x}\right)=\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial S}{\partial x}+q_2\right)-\frac{\partial}{\partial y}\left(\frac{\partial S}{\partial y}+q_1\right)+h\frac{\partial}{\partial y}(E\alpha\theta)-\right.$$
$$\left.-h\frac{\partial}{\partial x}\left(\frac{2\,(1+\mu)}{h}S\,(x,y)\right)\right),$$
$$\left(\frac{\partial^2}{\partial x\partial y}\ln h\right)(T_2-\mu T_1)+\frac{\partial}{\partial x}\ln h\left(\frac{\partial T_2}{\partial y}-\mu\frac{\partial T_1}{\partial y}\right)=\frac{\partial}{\partial y}\left(\mu\left(\frac{\partial S}{\partial y}+q_1\right)-\frac{\partial}{\partial x}\left(\frac{\partial S}{\partial x}+q_2\right)+h\frac{\partial}{\partial x}(E\alpha\theta)-\right.$$
$$\left.-h\frac{\partial}{\partial y}\left(\frac{2\,(1+\mu)}{h}S\,(x,y)\right)\right). \tag{14}$$

Relations (14) with account for (11)–(12) represent a system of two linear algebraic equations for $T_1$ and $T_2$ whose solution yields

$$T_1=\frac{1}{(1+\mu)\,\Delta}\,C\left(\frac{\partial^2\ln h}{\partial x\partial y}+\mu^2\frac{\partial\ln h}{\partial x}\frac{\partial\ln h}{\partial y}\right)+\mu D\left(\frac{\partial^2\ln h}{\partial x\partial y}+\mu^2\frac{\partial\ln h}{\partial x}\frac{\partial\ln h}{\partial y}\right),$$
$$T_2=\frac{1}{(1+\mu)\,\Delta}\,\mu C\left(\frac{\partial^2\ln h}{\partial x\partial y}+\mu^2\frac{\partial\ln h}{\partial x}\frac{\partial\ln h}{\partial y}\right)+D\left(\frac{\partial^2\ln h}{\partial x\partial y}+\mu^2\frac{\partial\ln h}{\partial x}\frac{\partial\ln h}{\partial y}\right), \tag{15}$$

where

$$\Delta=\left(\frac{\partial^2\ln h}{\partial x\partial y}\right)^2-\mu^2\left(\frac{\partial\ln h}{\partial x}\frac{\partial\ln h}{\partial y}\right)^2;$$
$$C=\frac{\partial\ln h}{\partial y}\left((1-\mu^2)\left(\frac{\partial S}{\partial y}+q_1\right)+\mu h\frac{\partial}{\partial y}\frac{2\,(1+\mu)\,S}{h}-\mu h\frac{\partial}{\partial x}(E\alpha\theta)\right)-\frac{\partial}{\partial y}\left(\frac{\partial S}{\partial y}+q_1\right)+$$
$$+\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial S}{\partial x}+q_2\right)+h\frac{\partial}{\partial y}(E\alpha\theta)-h\frac{\partial}{\partial x}\left(\frac{2\,(1+\mu)\,S}{h}\right)\right);$$
$$D=\frac{\partial\ln h}{\partial x}\left((1-\mu^2)\left(\frac{\partial S}{\partial x}+q_2\right)+\mu h\frac{\partial}{\partial x}\frac{2\,(1+\mu)\,S}{h}-\mu h\frac{\partial}{\partial y}(E\alpha\theta)\right)-\frac{\partial}{\partial x}\left(\frac{\partial S}{\partial x}+q_2\right)+$$
$$+\frac{\partial}{\partial y}\left(\mu\left(\frac{\partial S}{\partial y}+q_1\right)+h\frac{\partial}{\partial x}(E\alpha\theta)-h\frac{\partial}{\partial y}\left(\frac{2\,(1+\mu)\,S}{h}\right)\right).$$

Formulas (15) have been derived under the assumption that $\Delta\neq 0$.

Substituting expressions (15) obtained for $T_1$ and $T_2$ into the third equilibrium equation (3), we find the equation sought for determination of the geometric shape of the shell:

$$\frac{1}{R_1}\frac{1}{(1+\mu)\,\Delta}\left(C\left(\frac{\partial^2 \ln h}{\partial x \partial y}+\mu^2\frac{\partial \ln h}{\partial x}\frac{\partial \ln h}{\partial y}\right)+\mu D\left(\frac{\partial^2 \ln h}{\partial x \partial y}+\mu^2\frac{\partial \ln h}{\partial x}\frac{\partial \ln h}{\partial y}\right)\right)+$$

$$+\frac{1}{R_2}\frac{1}{(1+\mu)\,\Delta}\left(\mu C\left(\frac{\partial^2 \ln h}{\partial x \partial y}+\mu^2\frac{\partial \ln h}{\partial x}\frac{\partial \ln h}{\partial y}\right)+D\left(\frac{\partial^2 \ln h}{\partial x \partial y}+\mu^2\frac{\partial \ln h}{\partial x}\frac{\partial \ln h}{\partial y}\right)\right)=q_n\,. \tag{16}$$

Expression (16) is basic in solution of inverse problems of the theory of thin-walled thermoelastic shells. In solving them, part of the geometric parameters are prescribed, whereas formula (16) is used for determination of the remaining parameters.

## NOTATION

$A$ and $B$, coefficients of the first quadratic form of the median surface; $E$, Young modulus; $h(x, y)$, shell thickness; $q_1$, $q_2$, and $q_3$, external load; $1/R_1$, $1/R_2$, and $1/R_{12}$, curvatures and torsion of the median surface; $T_1$, $T_2$, and $S(x, y)$, generalized stretching and tangential forces acting in normal cross sections of the shell; $\alpha$ coefficient of thermoelasticity; $\mu$, Poisson coefficient; $\theta$, temperature field.

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